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A. A. MARKOV

On the Representation of Recursive Functions

[*Izvestiya Akademii Nauk SSSR. Seriya*

Matematicheskaya 13, 417-424 (1949)]

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On the Representation of Recursive Functions

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A. A. Марков

С представлением рекурсивных функций

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In this paper we shall establish a simple characterization of those primitive recursive functions P of one argument having the property that every general recursive function of n arguments can be represented in the form

$$P(\mu y(Q(x_1, \dots, x_n, y) = 0)),$$

where Q is a primitive recursive function of $n + 1$ arguments.

In 1936, Kleene (5) proved that every general recursive function of n arguments can be represented in the form

$$1.1 \quad P(\mu y(Q(x_1, \dots, x_n, y) = 0)),$$

where P is a primitive recursive function of one argument, and Q is a primitive recursive function of $n + 1$ arguments satisfying the condition

$$1.2 \quad (x_1) \dots (x_n) (Ey) (Q(x_1, \dots, x_n, y) = 0).$$

Here " $\mu y\dots$ " signifies the least of the numbers y which satisfy the condition following " μy ". The quantifiers* refer to the set of natural numbers $0, 1, 2, \dots$, which we shall simply call numbers.

Later, Kleene (4) refined his result, showing that it is possible to use a single fixed primitive recursive function P , independent of the general recursive function being represented. He designated the function which is universal in the above sense by P and defined it by an extremely complicated combination of schemes of substitution and primitive recursion. At the same time, Kleene extended the representation theorem to partial recursive functions.

In 1944, Skolem (8) suggested the possibility of doing without the function P , that is, of representing every general recursive function of n arguments in the form

$$\mu y(Q(x, y) = 0),$$

*As regards terminology and notation, we shall follow (1) and (2) with one change, namely, we shall use \equiv instead of \sim to denote equivalence. For brevity, we shall write x instead of x_1, \dots, x_n and (x) instead of $(x_1) \dots (x_n)$.

where Q is a primitive recursive function of $n + 1$ variables satisfying condition 1.2. He established a simple characterization of the functions admitting such a representation (9), leaving open the question of coincidence of the class of these functions with the class of general recursive functions.

This question was soon decided in the negative by Post (7). Skolem's proposition was thus refuted.

In connection with this, the following question naturally arose: For given n , what form must the primitive recursive function P of one argument take, in order that every general recursive function of n arguments may be represented in the form 1.1, using a suitable primitive recursive function Q of $n + 1$ arguments (dependent on the function represented) which satisfies condition 1.2?

The theorem of Kleene cited above shows that such functions exist, and from the aforementioned result of Post, it follows that not all primitive recursive functions of one argument are of this type. It is therefore natural to ask: What kind of functions are of this type?

The solution of this problem was indicated in a note of the author (6). The present article contains a detailed proof and some sharpening of that result*.

2. We shall say of an arithmetic function P of one argument, that it is a function of *large oscillation*, if it assumes every number as value an infinite number of times, that is if

$$(y)(z)(\exists x)(x > y \& P(x) = z),$$

We shall say of an arithmetic function P of one argument, that it is universal for n arguments, if it is primitive recursive, and every general recursive function of n arguments can be represented in the form 1.1, where Q is a primitive recursive function of $n + 1$ arguments, satisfying condition 1.2.

The following result holds:

Theorem 1. In order that a primitive recursive function of one argument be universal for n arguments, it is necessary and

*This result is presented below, in order to make the exposition of this paper independent of the aforementioned note.

sufficient that it be a function of large oscillation.

We observe that a primitive recursive function of large oscillation can be constructed easily. In particular, the functions σ_1 and σ_2 of Hilbert and Bernays have this property.* In this way, theorem 1 generalizes and improves Kleene's result, in regard to the outer function P of the representation 1.1.**

3. Theorem 1 is evidently a consequence of the following two theorems.

Theorem 2. If P is a primitive recursive function of large oscillation, then, for any partial recursive function R of n -arguments, there exists a primitive recursive function Q of $n+1$ arguments, such that

$$3.1 \quad R(x) \simeq P(\mu y(Q(x, y) = 0)).$$

Theorem 3. For any fixed integer n , a general recursive function of n arguments can be constructed with the property that for every representation thereof in the form 1.1 using primitive recursive functions P and Q , Q satisfying condition 1.2, P is a function of large oscillation.

The sign \simeq , occurring in theorem 2, denotes the coincidence of the functions defined by the expressions on the left and right of it. Speaking more precisely, we place this sign between two expressions defining arithmetic functions when we wish to affirm: firstly, that if either expression has meaning when its free variables assume specific numerical values, then the other expression has meaning when its free variables assume these same values; and secondly, that the values of these expressions are equal when their free variables assume any specific numerical values such that both expressions make sense.

4. In the lemma which we shall now formulate, we shall use an arithmetic function β of two variables defined by

*Vide (2), pp. 321 and 328. That σ_1 and σ_2 are functions of large oscillation follows directly from equations 6.42 and 6.43 given below.

**We do not consider explicitly here the normalization of the inner function Q established by Kleene. From our proof of theorem 2 (see below) it follows easily, however, that a normalization of the inner function, analogous to Kleene's, is possible for an arbitrary allowable choice of the outer function.

$$\beta(y, z) = \begin{cases} 0 & \text{for } y \neq z, \\ 1 & \text{for } y = z. \end{cases}$$

As is well known, this function is primitive recursive.*

Lemma. Let P be a primitive recursive function of large oscillation. We define a function W of two variables as follows:

$$4.2 \quad W(0, z) = 0,$$

$$4.21 \quad W(y', z) = W(y, z) + \beta(P(y), z);$$

the function V of one argument is given by the equation

$$4.22 \quad V(y) = W(y, P(y)).$$

Then V is a primitive recursive function, and for any binary predicate \mathfrak{U} whatsoever,^{**}

$$4.3 \quad (Ey)(Ez)\mathfrak{U}(y, z) \equiv (Et)\mathfrak{U}(V(t), P(t)).$$

Proof. In view of the primitive recursiveness of the function β and the hypothesized primitive recursiveness of P , we conclude from 4.2 and 4.21 that W is a primitive recursive function. From this it follows, by 4.22, that V is a primitive recursive function.

We define a function U of two arguments by the equation

$$4.4 \quad U(y, z) = \mu x(x \geq y \& P(x) = z);$$

the right side of 4.4 has meaning for all y and z , since P is a function of large amplitude.

The function U is general recursive, according to a theorem of Kleene, which asserts that the class of general recursive functions is closed with respect to the operator μ .*** This theorem is essential, for the proof we are giving to be constructive. We define a function T of two arguments by the primitive recursion

$$4.41 \quad T(0, z) = U(0, z),$$

$$4.42 \quad T(y', z) = U(T(y, z)', z).$$

*Vide, for example, (2), p. 317.

**All predicates considered here refer to the natural numbers.

***Vide (4), p. 51.

(The function T is also general recursive.)

We have:

$$\begin{aligned}
 4.5 & U(y, z) \geq y & [4.4] \\
 4.51 & P(U(y, z)) = z & [4.4] \\
 & y \leq x < U(y, z) \rightarrow P(x) \neq z & [4.4] \\
 4.52 & \rightarrow W(x', z) = W(x, z) & [4.21, 4.1] \\
 4.6 & W(U(y, z), z) = W(y, z) & [4.52, 4.5] \\
 & W(U(y, z)', z) = W(U(y, z), z)' & [4.21, 4.1, 4.51] \\
 4.61 & = W(y, z)' & [4.6] \\
 & W(T(0, z)', z) = W(0, z)' & [4.41, 4.61] \\
 4.7 & = 0' & [4.2] \\
 & W(T(y', z)', z) = W(U(T(y, z)', z)', z) & [4.42] \\
 4.71 & = W(T(y, z)', z)' & [4.61] \\
 4.72 & W(T(y, z)', z) = y' & [4.7, 4.71] \\
 & V(U(y, z)) = W(U(y, z), z) & [4.22, 4.51] \\
 4.8 & = W(y, z) & [4.6] \\
 4.81 & V(T(0, z)) = 0 & [4.41, 4.8, 4.2] \\
 & V(T(y', z)) = V(U(T(y, z)', z)) & [4.42] \\
 & = W(T(y, z)', z) & [4.8] \\
 4.82 & = y' & [4.72] \\
 4.83 & V(T(y, z)) = y & [4.81, 4.82] \\
 4.84 & P(T(y, z)) = z & [4.41, 4.42, 4.51]
 \end{aligned}$$

Now let \mathfrak{U} be any binary predicate. Then we have

$$\begin{aligned}
 4.9 & \mathfrak{U}(y, z) \rightarrow \mathfrak{U}(V(T(y, z)), P(T(y, z))) \\
 & \rightarrow (Et) \mathfrak{U}(V(t), P(t)) \\
 & (Ey)(Ez) \mathfrak{U}(y, z) \rightarrow (Et) \mathfrak{U}(V(t), P(t))
 \end{aligned}$$

Since the reverse implication is evident, we have 4.3, which was to be proved.

(In connection with this proof the following question arises:
Can the functions U and T be not primitive recursive, given the conditions of the lemma?)

5. We shall now prove theorem 2.

Let P be any primitive recursive function of large oscil-

lation and let R be any partial recursive function of n arguments.

According to Kleene's theorem on the representation of partial recursive functions,* there exist a primitive recursive function F of one argument, and an $(n+1)$ -ary primitive recursive predicate \mathfrak{B} such that

$$\begin{aligned} 5.1 \quad R(\xi) &\simeq F(\mu y \mathfrak{B}(\xi, y)), \\ 5.11 \quad (\xi)(y)(\mathfrak{B}(\xi, y) \rightarrow F(y)) &= R(\xi). \end{aligned}$$

Construct the functions W and V as in the lemma. According to the lemma, V is a primitive recursive function of one argument, and for an arbitrary binary predicate \mathfrak{U} , 4.3 is fulfilled.

Define the $(n+1)$ -ary predicate \mathfrak{C} by

$$5.2 \quad \mathfrak{C}(\xi, t) \equiv \mathfrak{B}(\xi, V(t)) \& P(t) = F(V(t)).$$

According to Gödel's theorems on primitive recursive functions and primitive recursive predicates,** the predicate \mathfrak{C} is primitive recursive, that is, there exists a primitive recursive function Q of $n + 1$ arguments such that

$$5.21 \quad \mathfrak{C}(\xi, t) \equiv Q(\xi, t) = 0.$$

We shall now prove that the representation 3.1 holds.

We have

$$\begin{aligned} 5.3 \quad (Ez)(z = F(y)) \\ 5.31 \quad \mathfrak{B}(\xi, y) \equiv \mathfrak{B}(\xi, y) \& (Ez)(z = F(y)) & [5.3] \\ \equiv (Ez)(\mathfrak{B}(\xi, y) \& z = F(y)) \\ (Ey)\mathfrak{B}(\xi, y) \equiv (Ey)(Ez)(\mathfrak{B}(\xi, y) \& z = F(y)) & [5.31] \\ \equiv (Et)\mathfrak{C}(\xi, t). & [4.3, 5.2] \end{aligned}$$

Because of 5.1, it follows from this that $R(\xi)$ has meaning if and only if $(Et)\mathfrak{C}(\xi, t)$, that is, when the expression $P(\mu^+\mathfrak{C}(\xi, t))$ makes sense.

In addition

$$\begin{aligned} 5.4 \quad \mathfrak{C}(\xi, t) &\rightarrow \mathfrak{B}(\xi, V(t)) \\ 5.41 \quad &\rightarrow F(V(t)) = R(\xi) & [5.41] \\ 5.42 \quad \mathfrak{C}(\xi, t) &\rightarrow P(t) = F(V(t)) & [5.2] \\ 5.42 \quad \mathfrak{C}(\xi, t) &\rightarrow P(t) = R(\xi). & [5.4, 5.41] \end{aligned}$$

*Vide (4), p.53

**Vide (3), p.180, theorems I-III.

If

$$P(\mu t \mathbb{C}(x, t))$$

has meaning, then

$$\mathbb{C}(x, \mu t \mathbb{C}(x, t)),$$

and therefore, by 5.42,

$$R(x) = P(\mu t \mathbb{C}(x, t)).$$

With this, it has been proved that

$$R(x) \sim P(\mu t \mathbb{C}(x, t)),$$

and according to 5.21, this yields 3.1.

6. In the proof of theorem 3, with which we now concern ourselves, an essential role is played by a general recursive, but not primitive recursive function of one argument, which takes only the values 0 and 1. A function with this property was constructed by Skolem (9).

Let ω be a general recursive function of one argument such that

6.1 ω is not a primitive recursive function,

$$6.11 \quad (x)(\omega(x) = 0 \vee \omega(x) = 1).$$

We must construct, for an arbitrary positive integer n , a general recursive function of n arguments such that, for every representation of it in form 1.1 with primitive recursive P and Q , Q satisfying 1.2, P is a function of large oscillation.

To begin with, we consider the case $n = 2$. We shall show that in this case the function ϕ_2 of two arguments defined by the equation

$$6.2 \quad \phi_2(x_1, x_2) = \omega(x_1) + x_2$$

possesses all the required properties.

Indeed, this function is general recursive since ω is a general recursive function. We assume that the representation

$$6.21 \quad \phi_2(x_1, x_2) = P(\mu y(Q(x_1, x_2, y) = 0))$$

holds, where P is a primitive recursive function of one argument and Q is a primitive recursive function of three arguments satisfying condition 1.2 with $n = 2$. We shall show that P is a function of large oscillation.

Assume the contrary. Then, in accordance with the definition of a function of large oscillation (vide paragraph 2), there exist numbers a and b such that

$$6.22 \quad (z)(P(z) = b \rightarrow z \leq a).$$

Choose such a pair of numbers a and b .

We define in succession the ternary predicate \mathfrak{D} , the ternary predicate \mathfrak{E} , and the unary predicate \mathfrak{F} by

$$6.23 \quad \mathfrak{D}(x_1, x_2, y) \equiv Q(x_1, x_2, y) = 0,$$

$$6.24 \quad \mathfrak{E}(x_1, x_2, y) \equiv \mathfrak{D}(x_1, x_2, y) \& (t)(t < y \rightarrow \mathfrak{D}(x_1, x_2, t)),$$

$$6.25 \quad \mathfrak{F}(x) \equiv (\exists z)(z \leq a \& P(z) = b \& \mathfrak{E}(x, b, z)).$$

The predicate \mathfrak{D} is primitive recursive, since the function Q is primitive recursive. In accordance with theorems of Gödel,* the predicates \mathfrak{E} and \mathfrak{F} are also primitive recursive. From this it follows that the function G defined by

$$6.26 \quad G(x) = \begin{cases} 0, & \text{if } \mathfrak{F}(x), \\ 1, & \text{if } \neg \mathfrak{F}(x), \end{cases}$$

is primitive recursive.

We have

$$6.27 \quad \begin{aligned} \mathfrak{E}(x_1, x_2, y) &\equiv y = \mu z \mathfrak{D}(x_1, x_2, z) & [6.24] \\ \omega(x) = 0 &\equiv \varphi_2(x, b) = b & [6.2] \\ &\equiv P(\mu y \mathfrak{D}(x, b, y)) = b & [6.21, 6.23] \end{aligned}$$

$$\equiv (\exists z)(z = \mu y \mathfrak{D}(x, b, y) \& P(z) = b)$$

$$\equiv (\exists z)(\mathfrak{E}(x, b, z) \& P(z) = b) & [6.27]$$

$$\equiv \mathfrak{F}(x) & [6.22, 6.25]$$

$$6.28 \quad \equiv G(x) = 0 & [6.26]$$

$$\omega(x) = 1 \equiv \omega(x) \neq 0 & [6.11]$$

$$\equiv G(x) \neq 0 & [6.28]$$

$$6.29 \quad \equiv G(x) = 1 & [6.26]$$

$$\omega(x) = G(x) & [6.11, 6.28, 6.29]$$

*Vide (3), p. 180, theorems I-IV.

Consequently, ω is a primitive recursive function, contrary to 6.1. This contradiction proves theorem 3 for $n = 2$.

We now consider the case $n > 2$. In this case we define a function ϕ_n of n arguments by the equation

$$6.3 \quad \phi_n(x_1, x_2, \dots, x_n) = \phi_2(x_1, x_2).$$

This function is general recursive, since ϕ_2 is a general recursive function.

Assume that the relation

$$6.31 \quad \varphi_n(x) = P(\mu y(Q(x, y) = 0))$$

holds, where P is a primitive recursive function of one argument and Q is a primitive recursive function of $n + 1$ arguments, satisfying condition 1.2. We shall show that P is a function of large oscillation.

We have

$$\varphi_2(x_1, x_2) = P(\mu y(Q(x_1, x_2, 0, \dots, 0, y) = 0)). \quad [6.3, 6.31]$$

Here the expression $Q(x_1, x_2, 0, \dots, 0, y)$ defines a primitive recursive function of three arguments x_1, x_2 , and y for which

$$(x_1)(x_2)(Ey)(Q(x_1, x_2, 0, \dots, 0, y) = 0).$$

Therefore, according to what was proved for $n = 2$, P is a function of large oscillation.

The case $n = 1$ remains. In this case we define a function ϕ_1 , of one argument by the equation

$$6.4 \quad \varphi_1(x) = \varphi_2(\sigma_1(x), \sigma_2(x)),$$

where σ_1 and σ_2 are the above mentioned primitive recursive functions of Hilbert and Bernays.* The function ϕ_1 is general recursive, since ϕ_2, σ_1 , and σ_2 are general recursive functions.

We assume that the representation

*Vide (2), pp. 321 and 328.

$$6.41 \quad \varphi_1(x) = P(\mu y(Q(x, y) = 0))$$

holds, where P is a primitive recursive function of one argument, and Q is a primitive recursive function of two arguments satisfying condition 1.2 with $n = 1$. We shall show that P is a function of large oscillation. In order to do this we shall make use of a primitive recursive function σ of two arguments, having the properties*

$$6.42 \quad \sigma_1(\sigma(x_1, x_2)) = x_1,$$

$$6.43 \quad \sigma_2(\sigma(x_1, x_2)) = x_2.$$

We have

$$\begin{aligned} \varphi_2(x_1, x_2) &= \varphi_2(\sigma_1(\sigma(x_1, x_2)), \sigma_2(\sigma(x_1, x_2))) & [6.42, 6.43] \\ &= \varphi_1(\sigma(x_1, x_2)) & [6.4] \\ &= P(\mu y(Q(\sigma(x_1, x_2), y) = 0)). & [6.44] \end{aligned}$$

Here the expression $Q(\sigma(x_1, x_2), y)$ defines a primitive recursive function of three arguments x_1, x_2 , and y , for which

$$(x_1)(x_2)(Ey)(Q(\sigma(x_1, x_2), y) = 0).$$

Therefore, in accordance with what was proved for $n = 2$, P is a function of large oscillation, which was to be proved.

This proof of theorem 3 is not constructive, inasmuch as it is a proof "by contradiction" and depends upon the law of the excluded middle. To replace this by a constructive proof hardly seems possible. However, the reasoning used furnishes a constructive proof of the following weakened form of theorem 3.

We shall say of an arithmetic function P of one argument, that it is a function of small oscillation if there exist numbers y and z , such that y is an upper bound for the numbers x which satisfy $P(x) = z$; that is, if

$$(Ey)(Ez)(x)(P(x) = z \rightarrow x \leq y).$$

A constructively provable form of theorem 3 is contained in the following:

For every positive integer n , a general recursive function

*Vide (2), p. 321.

of n arguments can be constructed, which does not admit representation of the form 1.1, with Q a primitive recursive function of $n + 1$ arguments satisfying 1.2, and P a primitive recursive function of small oscillation.

It is evident that this assertion contains the negative solution of the problem of Skolem formulated above.

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